Brander and Lewis Lecture

by

Richard D. MacMinn

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Uncertainty, Duopoly, and Finance

I. The Financial Market Model

Consider an economy composed of product and financial markets in which all agents interacting in the markets are risk averse. Suppose two corporations operate in an imperfectly competitive product market and finance their operations in a competitive financial market. Suppose the product market is characterized by a Cournot duopoly. Let $\Pi_f(q_1, q_2, \sigma)$, f = 1, 2, be corporation f's payoff in the product market, where $\sigma \quad S \equiv [0, \omega]$ denotes a state of nature and S denotes the set of states of nature that can occur then. Given a Cournot duopoly, the payoff of corporation number one may be characterized as

$$\Pi_1(q_1, q_2, \sigma) = h(q_1 + q_2, \sigma)\, q_1 - c_1(q_1)$$

where h is the random market demand function. The payoff of corporation two is characterized similarly. Suppose the random demand function satisfies the Principle of Increasing Uncertainty (PIU) (Leland 1972; MacMinn and Holtmann 1983) so that $\partial h / \partial \sigma > 0$ and $\partial^2 h / \partial q_f \partial \sigma > 0$ for f = 1, 2. It follows that the payoff functions also satisfy the PIU since

$$\frac{\partial \Pi_1}{\partial \sigma} = \frac{\partial h}{\partial \sigma} q_1 > 0 \text{ and } \frac{\partial^2 \Pi_1}{\partial q_1 \partial \sigma} = \frac{\partial h}{\partial \sigma} + \frac{\partial^2 h}{\partial q_1 \partial \sigma} q_1 > 0$$

These, of course, are simply the conditions that revenue and marginal revenue are increasing in state.

Each corporation also operates in the financial market. It is supposed here that all investors are risk averse and transfer money from now to then or then to now by selecting portfolios in the financial market. It is assumed here that the financial market is complete. In such a setting it is possible to define stock that pay off one dollar then in one state σ and zero otherwise (Arrow 1963). Call these contracts the basis stock. Then it is apparent that all contracts can be expressed as a portfolio of basis stock. Letting p(σ) denote the price of a basis stock now, it follows that the value of the common stock of a levered firm with a promised payment of b_f then on its bonds is S_f, where

$$S_f = \int_S p(\sigma)\ \max\left\{0, \Pi_f(q, \sigma) - b_f\right\}\ d\sigma = \int_{\zeta_f}^{\omega} p(\sigma)\left[\Pi_f(q, \sigma) - b_f\right] d\sigma$$

where ζ_f is the boundary of the solvency event and implicitly defined by the condition $\Pi_f(q, \zeta_f) - b_f = 0$. It should be observed that the boundary of the solvency event, i.e., ζ_f, is a function of q and b_f. Similarly, the value of the firm's bond issue is D_f, where

$$D_f = \int_S p(\sigma)\ \min\left\{\Pi_f(q, \sigma), b_f\right\}\ d\sigma = \int_0^{\zeta_f} p(\sigma)\ \Pi_f(q, \sigma)\ d\sigma + \int_{\zeta_f}^{\omega} p(\sigma)\ b_f\ d\sigma$$

Assume that the firms have obtained the necessary funds in the financial market and that each determines an optimal output in the product market by maximizing levered stock value. In a Cournot duopoly each firm takes the output of the other as given and in equilibrium this assumption is satisfied. The first and second order conditions are

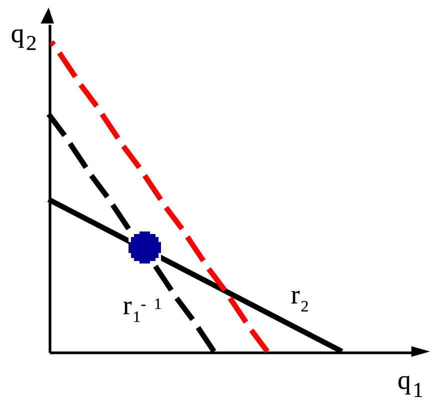
$$\frac{\partial S_f}{\partial q_f} = \int_{\zeta_f}^{\omega} p(\sigma)\ \frac{\partial \Pi_f}{\partial q_f}\ d\sigma = 0 \qquad \text{(FOC)}$$

$$\frac{\partial^2 S_f}{\partial^2 q_f} = -\ p(\zeta_f)\ \frac{\partial \Pi_f}{\partial q_f}\ \frac{\partial \zeta_f}{\partial q_f} + \int_{\zeta_f}^{\omega} p(\sigma)\ \frac{\partial^2 \Pi_f}{\partial^2 q_f}\ d\sigma < 0 \qquad \text{(SOC)}$$

for f = 1, 2. The simulataneous solution of the conditions in (FOC) is the Cournot duopoly solution. A Cournot solution also requires the following two conditions:

$$\frac{\partial S_f}{\partial q_f \, \partial q_g} < 0, \text{ for } f, g = 1, 2 \qquad (1)$$

$$\frac{\partial^2 S_1}{\partial q_1 \, \partial q_1} \frac{\partial^2 S_2}{\partial q_2 \, \partial q_2} - \frac{\partial^2 S_1}{\partial q_1 \, \partial q_2} \frac{\partial^2 S_2}{\partial q_2 \, \partial q_1} > 0 \qquad (2)$$



Condition (1) guarantees that the reaction functions are decreasing and condition (2) guarantees that the equilibrium is stable. These conditions are demonstrated in the figure.

Alternatively, one may derive the reaction functions. Note that

$$\frac{\partial S_1(q_1, q_2)}{\partial q_1} = \int_{\zeta_1}^{\omega} p(\sigma) \frac{\partial \Pi_1}{\partial q_1} \, d\sigma \qquad (R)$$

and if the second order condition is satisfied then it follows by the Implicit Function Theorem that there exists a differentiable function r_1 such that $q_1 = r_1(b_1, q_2)$, equation (R) equals zero[1] and

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[1] The reaction function must satisfy the condition

$$\frac{\partial S_1(b_1, \ r_1(b_1, \ q_2), \ q_2)}{\partial q_1} = 0$$

Setting the differential equal to zero and rearranging yields

$$dq_1 = -\frac{\dfrac{\partial^2 S_1}{\partial q_1 \, \partial b_1}}{\dfrac{\partial^2 S_1}{\partial^2 q_1}} \, db_1 - \frac{\dfrac{\partial^2 S_1}{\partial q_1 \, \partial q_2}}{\dfrac{\partial^2 S_1}{\partial^2 q_1}} \, dq_2$$

and the partial derivatives of the reaction function are based on this result.

$$\frac{\partial r_1}{\partial q_2} = - \frac{\dfrac{\partial^2 S_1}{\partial q_2\, \partial q_1}}{\dfrac{\partial^2 S_1}{\partial^2 q_1}}$$

Then duopolist one decreases output as duopolist two increases output if the numerator in this derivative is negative. The sign of the numerator is not obvious as the following derivative shows

$$\frac{\partial^2 S_1}{\partial q_1\, \partial q_2} = - p(\zeta_1) \frac{\partial \Pi_1}{\partial q_1} \frac{\partial \zeta_1}{\partial q_2} + \int_{\zeta_1}^{\omega} p(\sigma) \frac{\partial^2 \Pi_1}{\partial q_1\, \partial q_2}\, d\sigma$$

Of course, BL (Brander and Lewis 1986) assume that the numerator is negative. To see whether such an assumption is reasonable, we must determine the sign of $\partial\zeta_1/\partial q_2$ and $\partial^2\Pi_1/\partial q_1\, \partial q_2$. A negative sign for the cross partial of the payoff function simply says that the marginal revenue of firm one is decreasing in the output of firm two.

Similarly, observe how the reaction function of this duopolist changes with the leverage b_1. Note that

$$\frac{\partial r_1}{\partial b_1} = - \frac{\dfrac{\partial^2 S_1}{\partial q_1\, \partial b_1}}{\dfrac{\partial^2 S_1}{\partial^2 q_1}}$$

Hence, the reaction function is increasing in b_1 if and only if the numerator is positive. Note that

$$\frac{\partial^2 S_1}{\partial q_1\, \partial b_1} = - p(\zeta_1) \frac{\partial \Pi_1}{\partial q_1} \frac{\partial \zeta_1}{\partial b_1} > 0 \qquad (3)$$

since $\partial\Pi_1 / q_1$ is negative at ζ_1 by the PIU and ζ_1 is an increasing function of b_1 since implicit differentiation of $\Pi_1(q, \zeta_1) - b_1 = 0$ yields

$$\frac{\partial\Pi_1}{\partial\zeta_1}\frac{\partial\zeta_1}{\partial b_1} - 1 = 0$$

or equivalently

$$\frac{\partial\zeta_1}{\partial b_1} = \frac{1}{\frac{\partial\Pi_1}{\partial\zeta_1}} > 0$$

These results combine to show that the reaction function of the duopolist is increasing in leverage, i.e., $\partial r_1 / \partial b_1 > 0$, as long as the debt issue is subject to some default risk. This result is demonstrated in the above figure. Of course, it is also apparent that the reaction function is not increasing in leverage as long as the probability of default is zero.

Consider the comparative statics associated with an increase in the leverage of firm one. The differential of the system of equations in (FOC) may be stated as

$$\frac{\partial^2 S_1}{\partial q_1^2} dq_1 + \frac{\partial^2 S_1}{\partial q_1\, \partial q_2} dq_2 + \frac{\partial^2 S_1}{\partial q_1\, \partial b_1} db_1 = 0$$

$$\frac{\partial^2 S_2}{\partial q_2\, \partial q_1} dq_1 + \frac{\partial^2 S_1}{\partial q_2^2} dq_2 + \frac{\partial^2 S_2}{\partial q_2\, \partial b_1} db_1 = 0$$

or in matrix form as

$$\begin{bmatrix} \dfrac{\partial^2 S_1}{\partial q_1^2} & \dfrac{\partial^2 S_1}{\partial q_1\,\partial q_2} \\ \dfrac{\partial^2 S_2}{\partial q_2\,\partial q_1} & \dfrac{\partial^2 S_2}{\partial q_2^2} \end{bmatrix} \begin{pmatrix} dq_1 \\ dq_2 \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial^2 S_1}{\partial q_1\,\partial b_1} \\ \dfrac{\partial^2 S_2}{\partial q_2\,\partial b_1} \end{pmatrix} db_1$$

Let

$$\left|H\right| \equiv \begin{vmatrix} \dfrac{\partial^2 S_1}{\partial q_1^2} & \dfrac{\partial^2 S_1}{\partial q_1\,\partial q_2} \\ \dfrac{\partial^2 S_2}{\partial q_2\,\partial q_1} & \dfrac{\partial^2 S_2}{\partial q_2^2} \end{vmatrix} > 0 \qquad (4)$$

The inequality in (4) follows by (2), i.e., by the stability condition. Then by Cramer's Rule

$$\frac{dq_1}{db_1} = \frac{\begin{vmatrix} -\dfrac{\partial^2 S_1}{\partial q_1\,\partial b_1} & \dfrac{\partial^2 S_1}{\partial q_1\,\partial q_2} \\ -\dfrac{\partial^2 S_2}{\partial q_2\,\partial b_1} & \dfrac{\partial^2 S_2}{\partial q_2^2} \end{vmatrix}}{\left|H\right|} = -\frac{\dfrac{\partial^2 S_1}{\partial q_1\,\partial b_1}\,\dfrac{\partial^2 S_2}{\partial q_2^2}}{\left|H\right|} > 0 \qquad (5)$$

and

$$\frac{dq_2}{db_1} = \frac{\begin{vmatrix} \dfrac{\partial^2 S_1}{\partial q_1^2} & -\dfrac{\partial^2 S_1}{\partial q_1\,\partial b_1} \\ \dfrac{\partial^2 S_2}{\partial q_2\,\partial q_1} & -\dfrac{\partial^2 S_2}{\partial q_2\,\partial b_1} \end{vmatrix}}{\left|H\right|} = \frac{\dfrac{\partial^2 S_2}{\partial q_2\,\partial q_1}\,\dfrac{\partial^2 S_1}{\partial q_1\,\partial b_1}}{\left|H\right|} < 0 \qquad (6)$$

The inequality in (5) follows by (SOC), (2), and (3). Similarly, the inequality in (6) follows by (1), (2), and (3).

II. The Financing Decision

Next, consider the financing decisions. BL talk about debt as a strategic commitment device. Whether there is a kernel of truth in this interpretation remains to be seen. BL also disallow the use of bond covenants. What is more, BL do not mention the rather well known agency problems associated with risky bond issues. This may explain why BL switch the objective function from stock value to corporate value in considering the financing decisions. I want to take a different approach and maintain the same objective function for each step in the decision making process.

Consider the sequence of decisions. At date zero, the corporate managers must select the financing provisions such that I dollars are raised. Then the corporate managers select a production level. The production level cannot be observed by bondholders at the time the bonds are issued but bondholders are rational and so understand the nature of the incentives management faces as well as the nature of the competition in the duopoly product market. Hence, for any pair of promised payments $b = (b_1, b_2)$, the bondholders can anticipate the production levels accurately. Suppose that the corporate managers determine the corporate capital structure. This may be done by supposing that the managers can issue any combination of new equity and new debt. It will be assumed here that both firms are initially unlevered. The financing condition facing each manager is of the form $S_f^n + D_f^n = I$, where the LHS of the financing condition represents the value of the new equity issue and new debt issue, respectively. Given any b the bondholders correctly anticipate q and value the debt issue appropriately. Since the promised payment implies the production level that satisfies the FOC, the manager has no incentive to attempt to shift risk. Of course, because of this rational anticipation of production level it may not be the case that the manager can raise the required funds with just a debt issue and so an equity issue may be required as well. When the manager makes the capital structure decision the problem becomes one in which the issues are made knowing that the bondholders know the consequences of the capital structure choice. The same thing must be said about any new equity holders.

Observe that the value of the new equity and debt issues may be expressed as

$$S_f^n = \int_0^{\omega} p(\sigma) \frac{n_f}{N_f + n_f} \max\left\{0, \Pi_f(q, \sigma) - b_f\right\} d\sigma$$

$$= \int_{\zeta_f}^{\omega} p(\sigma)\, a_f \left[\Pi_f(q, \sigma) - b_f\right] d\sigma$$

where n_f is the size of the new equity issue, or equivalently, the number of new shares and $a_f \equiv n_f / (N_f + n_f)$ is the proportion of the equity acquired by the new shareholders. Similarly, the value of the new debt issue is

$$D_f^n = \int_0^{\omega} p(\sigma) \min\left\{\Pi_f(q, \sigma), b_f\right\} d\sigma = \int_0^{\zeta_f} p(\sigma)\, \Pi_f(q, \sigma)\, d\sigma + \int_{\zeta_f}^{\omega} p(\sigma)\, b_f\, d\sigma$$

Since the firms are assumed to be initially unlevered D_f will be used subsequently for the value of the new debt issue. Having specified the financing condition it should follow that any manager with interests aligned to those of the shareholders selects the capital structure of the firm to

$$\text{maximize } S_f^o$$
$$\text{subject to } S_f^n + D_f = I \qquad (6)$$

where S_f^o represents the old shareholders' stake in the corporation and

$$S_f^o = \frac{N_f}{N_f + n_f} S_f = (1 - a_f)\, S_f$$

After the capital sturcture is selected, i.e., the requisite financing is done, the manager selects the production level again to maximize S_f^o but this time unconstrained. Hence, it should be noted that the manager's objective function remains the same in the sequence of decisions. Of course, it should also be noted that the production decision does not depend on the new equity holders stake a_f.

Now consider the manager's capital structure decision. The constrained maximization problem in (6) may be equivalently expressed in Lagrange form as

$$\text{maximize } L(a_f, b_f, \lambda)$$

where $L = S_f^o + \lambda\,(S_f^n + D_f - I)$. The conditions for a maximum are as follows:

$$\frac{\partial L}{\partial a_f} = -\,S_f + \lambda\,S_f = 0 \tag{7}$$

$$\frac{\partial L}{\partial b_f} = (1 - a_f)\,\frac{\partial S_f}{\partial b_f} + \lambda\left[a_f\,\frac{\partial S_f}{\partial b_f} + \frac{\partial D_f}{\partial b_f}\right] = 0 \tag{8}$$

where $V_f \equiv S_f + D_f$ represents the corporate value. By (7), condition (8) may be expressed as

$$\frac{\partial S_f}{\partial b_f} + \frac{\partial D_f}{\partial b_f} = \frac{\partial V_f}{\partial b_f} = 0 \tag{9}$$

Note that

$$\frac{\partial S_f}{\partial b_f} = -\ p(\zeta_f)\left[\Pi_f(q,\zeta_f) - b_f\right]\left(\frac{\partial \zeta_f}{\partial b_f} + \frac{\partial \zeta_f}{\partial q_f}\frac{\partial q_f}{\partial b_f} + \frac{\partial \zeta_f}{\partial q_g}\frac{\partial q_g}{\partial b_f}\right)$$

$$+ \int_{\zeta_f}^{\omega} p(\sigma)\left[\frac{\partial \Pi_f}{\partial q_f}\frac{\partial q_f}{\partial b_f} + \frac{\partial \Pi_f}{\partial q_g}\frac{\partial q_g}{\partial b_f}\right] d\sigma - \int_{\zeta_f}^{\omega} p(\sigma)\, d\sigma$$

$$= \frac{\partial q_g}{\partial b_f}\int_{\zeta_f}^{\omega} p(\sigma)\frac{\partial \Pi_f}{\partial q_g}\, d\sigma - \int_{\zeta_f}^{\omega} p(\sigma)\, d\sigma \qquad (10)$$

The second equality follows due to the definition of the state ζ_f and the first order condition for the subsequent production decision. Similarly,

$$\frac{\partial D_f}{\partial b_f} = p(\zeta_f)\left[\Pi_f(q,\zeta_f) - b_f\right]\left(\frac{\partial \zeta_f}{\partial b_f} + \frac{\partial \zeta_f}{\partial q_f}\frac{\partial q_f}{\partial b_f} + \frac{\partial \zeta_f}{\partial q_g}\frac{\partial q_g}{\partial b_f}\right)$$

$$+ \int_{0}^{\zeta_f} p(\sigma)\left[\frac{\partial \Pi_f}{\partial q_f}\frac{\partial q_f}{\partial b_f} + \frac{\partial \Pi_f}{\partial q_g}\frac{\partial q_g}{\partial b_f}\right] d\sigma + \int_{\zeta_f}^{\omega} p(\sigma)\, d\sigma$$

$$= \int_{0}^{\zeta_f} p(\sigma)\left[\frac{\partial \Pi_f}{\partial q_f}\frac{\partial q_f}{\partial b_f} + \frac{\partial \Pi_f}{\partial q_g}\frac{\partial q_g}{\partial b_f}\right] d\sigma + \int_{\zeta_f}^{\omega} p(\sigma)\, d\sigma \qquad (11)$$

Using (10) and (11), the derivative on the LHS of (9) may be rewritten as

$$\frac{\partial S_f}{\partial b_f} + \frac{\partial D_f}{\partial b_f} = \frac{\partial q_g}{\partial b_f}\int_{\zeta_f}^{\omega} p(\sigma)\frac{\partial \Pi_f}{\partial q_g}\, d\sigma - \int_{\zeta_f}^{\omega} p(\sigma)\, d\sigma$$

$$+ \int_{0}^{\zeta_f} p(\sigma)\left[\frac{\partial \Pi_f}{\partial q_f}\frac{\partial q_f}{\partial b_f} + \frac{\partial \Pi_f}{\partial q_g}\frac{\partial q_g}{\partial b_f}\right] d\sigma + \int_{\zeta_f}^{\omega} p(\sigma)\, d\sigma$$

$$= \frac{\partial q_g}{\partial b_f} \int_0^{\omega} p(\sigma) \frac{\partial \Pi_f}{\partial q_g} d\sigma + \frac{\partial q_f}{\partial b_f} \int_0^{\zeta_f} p(\sigma) \frac{\partial \Pi_f}{\partial q_f} d\sigma \qquad (12)$$

The second term on the RHS of (12) is negative because the PIU yields $\partial \Pi_f / \partial q_f < 0$ for all states σ $(0, \zeta_f) \equiv B_f$ and because $\partial q_f / \partial b_f > 0$ by (5). The first term on the RHS of (12), however, is positive because

$$\frac{\partial \Pi_f}{\partial q_g} = \frac{\partial h}{\partial q_g} q_f < 0$$

and $\partial q_g / \partial b_f < 0$ by (6). The two effects shown in (12) may be easily interpreted. First, the negative term is the marginal agency cost of leverage. Recall that increased leverage increases the manager's incentive to shift risk to the bondholders by increasing the output; bondholders recognize this and require a larger promised payment to compensate for it. Hence, there is an agency cost associated with a risky bond issue. Second, the positive term is the marginal benefit due to the competitor's reduced output, i.e., the gains due to the competitive advantage associated with leverage. Although the particular debt level is not clear this analysis does show that some risky debt is optimal. To see this simply suppose that no risky debt is issued. Then $\zeta_f \equiv 0$ and the second term on the RHS of (12) disappears, leaving a positive derivative. It should also be observed that the positive term on the RHS of (12) depends on the nature of the imperfect competition in the product market. If the product market is perfectly competitive then the positive term disappears and so the benefits of risky debt also disappear leaving only the marginal agency cost. Hence, a firm operating in a competitive product market has an incentive to create a capital structure that eliminates the risk of insolvency. A similar statement can be made for monopoly.

There is something very disturbing about this analysis. At one point in the sequence of decisions the corporate manager is supposed to act as a Cournot duopolist, i.e., in making the production decision. This means that the manager assumes that the other firms production decision does not change when the manager makes his or her production choice. At an earlier point in the decision sequence, however, the manager makes a capital structure decision and quite specifically takes the competitor's output effects due to leverage into account!

References

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