Capital Budgeting Problems

by

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Case I: The Wine Aging Problem

Suppose that an investment project requires a capital outlay of I_0 dollars now and returns v_t dollars at date t. The problem is to select the optimal time period for the investment. The time period is extended as long at the present value of the investment is increasing, i.e.,

$$\frac{v_T}{(1+r)^T} > \frac{v_{T-1}}{(1+r)^{T-1}}$$

$$\Leftrightarrow \frac{1}{(1+r)^T}\left\{v_T - (1+r)\,v_{T-1}\right\} > 0$$

$$v_T > (1+r)v_{T-1}. \tag{1}$$

The interpretation of this condition is simply that the value of the sales at date T exceed the future value of the sales at date T - 1, or equivalently, exceed the value of selling at date T - 1 and investing the procedes in the financial market. Notice that (1) may also be equivalently expressed as

$$\frac{v_T}{v_{T-1}} - 1 > r \tag{2}$$

The left hand side of (2) is the intertemporal internal rate of return. If this IRR exceeds the financial market rate then it is optimal to continue allowing the wine to age. Therefore, the optimal production period is that T such that

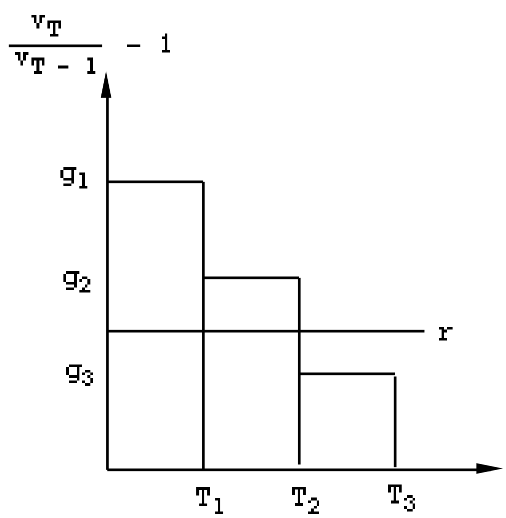
$$\frac{v_T}{v_{T-1}} - 1 > r \text{ and } \frac{v_{T+1}}{v_T} - 1 < r$$

As an example, suppose the value of the wine increases at the constant rate g_1 over the period $[0, T_1]$, at the rate g_2 over the period $(T_1, T_2]$, . . . , etc.. Observe that

$$\frac{v_T}{v_{T-1}} - 1 = \begin{cases} g_1 & 0 \le t \le T_1 \\ g_2 & T_1 < t \le T_2 \\ g_3 & T_2 < t \le T_3 \\ \vdots & \end{cases}$$

Then, the condition for an optimal time period of production can be rewritten as the T_i such that $g_i > r > g_{i+1}$. This is shown in the following figure where the optimal time period is T_2.

Figure: The Optimal Time Period of Production



Case II: The Machine Selection Problem

Consider two machines which perform the same tasks with equal efficiency but which have different costs and lives. Let c_i, m_i, and T_i denote the capital cost, operating cost, and life of machine i = A, B. The problem is to determine which machine to purchase.

There are two methods of solution. The first method is to construct an infinite chain of type A and B machines. Determine the present value of each chain and then select the machine with the smaller present value. The present value of an infinite chain of machines with lives of length T_i is

$$pv = \frac{m_i}{r} + c_i + \frac{c_i}{(1+r)^{T_i}} + \frac{c_i}{(1+r)^{2T_i}} + \ldots$$

$$= \frac{m_i}{r} + c_i \sum_{t=0}^{\infty} \frac{1}{(1+r)^{tT_i}}$$

$$= \frac{m_i}{r} + \frac{c_i}{r\, d_{T_i}}$$

since

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^{tT_i}} = 1 + \frac{1}{(1+r)^{T_i}} + \frac{1}{(1+r)^{2T_i}} + \ldots$$

$$\frac{1}{(1+r)^{T_i}} \sum_{t=0}^{\infty} \frac{1}{(1+r)^{tT_i}} = \frac{1}{(1+r)^{T_i}} + \frac{1}{(1+r)^{2T_i}} + \ldots$$

and subtracting yields

$$\left\{ 1 - \frac{1}{(1+r)^{T_i}} \right\} \sum_{t=0}^{\infty} \frac{1}{(1+r)^{tT_i}} = 1$$

$$\Leftrightarrow \sum_{t=0}^{\infty} \frac{1}{(1+r)^{tT_i}} = \frac{1}{1 - \frac{1}{(1+r)^{T_i}}} = \frac{1}{r\left(\frac{1}{r} - \frac{1}{r(1+r)^{T_i}} \right)} = \frac{1}{r\, d_{T_i}}$$

Therefore, machine A is selected if

$$\frac{m_A}{r} + \frac{c_A}{r\, d_{T_A}} < \frac{m_B}{r} + \frac{c_B}{r\, d_{T_B}}$$

$$\Leftrightarrow m_A + \frac{c_A}{d_{T_A}} < m_B + \frac{c_B}{d_{T_B}}$$

$$\Leftrightarrow \frac{c_A}{d_{T_A}} - \frac{c_B}{d_{T_B}} < m_B - m_A$$

The LHS is the difference in the annualized capital cost and the RHS is the difference in the operating costs.

The second method is to determine the annualized cost of each machine. The idea here is to generate an annual cost figure. To generate such a cost set the present value of an annuity with an unknown cash payment equal to the present value of one machine of type i = A, B, and then solve for the unknown cash payment. Letting p_i denote the cash payoff on an annuity of type i = A, B, set

$$p_i \, d_{T_i} = c_i + m_i \, d_{T_i}$$

It follows that

$$p_i = \frac{c_i}{d_{T_i}} + m_i$$

Since the operating costs are already annual cost, only the capital cost of the machine needs to be annualized. Now, select machine A over B if $p_A < p_B$, or equivalently, if

$$\frac{c_A}{d_{T_A}} + m_A < \frac{c_B}{d_{T_B}} + m_B$$

$$\Leftrightarrow \frac{c_A}{d_{T_A}} - \frac{c_B}{d_{T_B}} < m_B - m_A$$

Note, of course, that this is the same decision rule found using the previous method.

Case III: The Salvage Value Problem

Let R_t be the quasi-rent generated by a machine in year t and let S_T be the salvage value of the machine. The present value of operating the machine for T years is

$$pv_T = \frac{R_1}{1+r} + \ldots + \frac{R_T}{1+r} + \frac{S_T}{(1+r)^T}$$

The decision to operate the machine for another year is made if the present value increases, i.e., $pv_{T+1} > pv_T$. Note that

$$pv_{T+1} - pv_T = \sum_{t=1}^{T+1} \frac{R_t}{(1+r)^t} - \sum_{t=1}^{T} \frac{R_t}{(1+r)^t} + \frac{S_{T+1}}{(1+r)^{T+1}} - \frac{S_T}{(1+r)^T}$$

$$= \frac{R_{T+1}}{(1+r)^{T+1}} + \frac{S_{T+1}}{(1+r)^{T+1}} - \frac{S_T}{(1+r)^T} > 0$$

$$\frac{1}{(1+r)^{T+1}}\left[R_{T+1} + S_{T+1} - (1+r)\ S_T\right] > 0$$

$$\Leftrightarrow\ R_{T+1} + S_{T+1} > (1+r)\ S_T$$

$$\Leftrightarrow\ \frac{R_{T+1} + S_{T+1}}{S_T} > (1+r)$$

$$\Leftrightarrow\ \frac{R_{T+1} + S_{T+1} - S_T}{S_T} > r$$