The Fisher Model

and the

Foundations of the Net Present Value Rule

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Introduction

Financial markets perform of role of allowing individuals and corporations to transfer money between dates. The individual may save by transferring dollars from the present to the future. The corporation may invest and finance the investment by transferring dollars from the future to the present. The following model provides the theoretical foundation for the net present value rule and by extension the corporate objective function. It also demonstrates why we look to the financial markets to find the cost of capital.

In its simplest setting, this model of individual behavior incorporates only one time period and does not include uncertainty. The model is developed in three cases. First, allowing the consumer to participate in the financial market, the individual's savings decision is characterized.[1] Second, allowing the consumer make a savings decision on personal account and to be a firm proprietor who makes an investment decision on firm, the individual's saving/investment decision is characterized; in this case the savings and investment decision are tied together, i.e. what is saved on personal account is invested on firm account.[2] This investment decision is in a project, i.e., not a financial asset. Third, allowing the proprietor to make an investment decision and to participate in the financial market, the individual's savings and investment decisions are characterized, i.e., in this case the two decisions are not tied together and the individual make increase or decrease the saving relative to the investment. In each

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[1] Since there is no uncertainty, all financial assets must yield the same rate of return. Hence it is logical to suppose that there is only one financial asset and market. This will change when uncertainty is introduced.
[2] Note that the discount factor

$$\frac{1}{1 + r}$$

setting, the individual is assumed to behave in accordance with her own self-interest. It is important to note that we are not assuming that the individual makes any decisions to maximize net present value. If any model is to demonstrate the importance of the net present value rule then that model must show that the individual finds it optimal to use the rule. This result is demonstrated in case three.

Case I: The Savings Decision

A consumer's savings decision amounts to choosing how to allocate consumption between time periods. A simple model of this is as follows. Suppose that there is just one good and two time periods. The consumer receives an income of m_0 at date zero and m_1 at date one. Let c_0 be his dollar consumption at date zero, equivalently, *now*, and let c_1 be his dollar consumption at date one, equivalently, *then*. Suppose that he can borrow and lend at the interest rate r. The individual's savings now is $s_0 = m_0 - c_0$. This saving will grow to the value $(m_0 - c_0)(1 + r)$ *then*. Hence, consumption *then* equals this value plus income *then*, i.e.,

$$c_1 = m_1 + (m_0 - c_0)(1 + r)$$

Equivalently,

$$c_0 (1 + r) + c_1 = m_0 (1 + r) + m_1$$

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is simply the price now of a one dollar payoff then.

This simply says that the future value of consumption equals the future value of income. The value of consumption and income may be expressed in present value terms as

$$c_0 + \frac{c_1}{1 + r} = m_0 + \frac{m_1}{1 + r} \tag{B}$$

The left hand side is the present value of the consumption expenditure while the right hand side is the present value of the income stream. Equation (B) is the individual's intertemporal budget constraint.

In the next figure, consumption now is measured on the horizontal axis and consumption then is measured on the vertical axis. The "intertemporal budget line" B represents all combinations of consumption pairs (c_0, c_1) which have a present value equal to the consumer's wealth. The point m = (m_0, m_1) represents the time pattern of the consumer's income and, of course, lies on B, since the consumer could certainly satisfy his budget constraint by consuming each period's income in that period. B has a slope with an absolute value of (1 + r) corresponding to the increase in consumption then that would be possible if he gave up a unit of consumption now.[3] Note that (1 + r) is an opportunity cost since it is the cost of consumption now in terms of consumption then.

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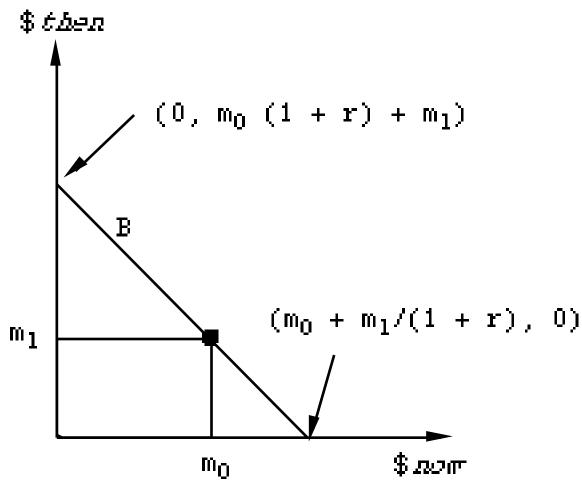
[3] Rearranging (B) so that it is in slope\intercept form yields

$$c_1 = ((1 + r)\, m_0 + m_1) - (1 + r)\, c_0,$$

where $(1 + r)\, m_0 + m_1$ is the intercept and $-(1 + r)$ is the slope.

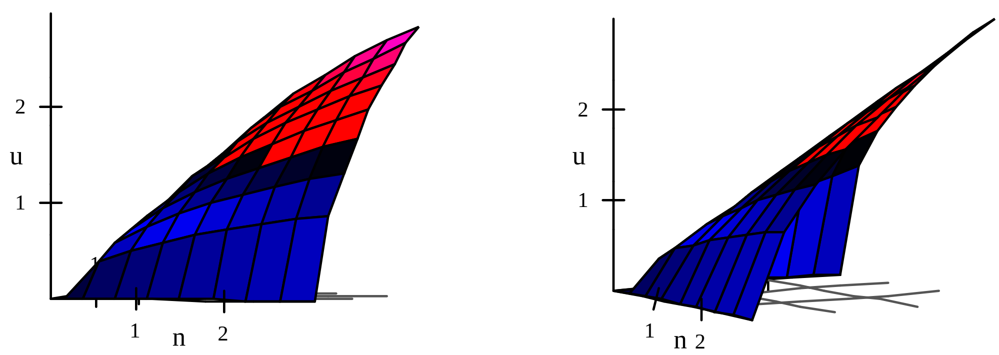
figure 1: the budget constraint



A greater income either now or then yields a higher budget line through the new income pair. A greater interest rate yields a steeper budget line since giving up a unit of consumption now would permit even more consumption then.

Next, consider the individual's preferences for consumption now versus then. These preferences are represented by the individual's utility function $u(c_0, c_1)$ which maps consumption pairs into real numbers, i.e. the larger the number the better the consumption pair. The consumer is assumed to prefer more to less and so utility increases with more consumption now and then. Some statements regarding the utility function are generic while others are not. Since the notion that more is preferred to less, the statement that utility increases in consumption applies generally. The utility function, however, also contains information concerning the individual's preference for more consumption now versus then and this preference is quite individual specific. The next figure presents two views of the same utility function.

figure 2: two views of the utility function[4]

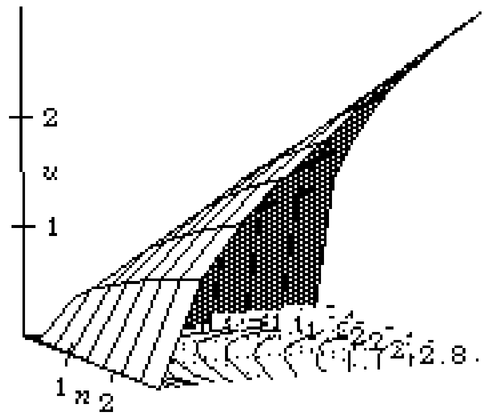


The contour lines of this utility function are shown in the next figure. Just as contour lines on a map connect the points on a land surface that have the same elevation, the contour lines in this figure connect the

[4] The axes are labeled n, t, u for dollar consumption now, dollar consumption then, and the utility number. The t label is hidden by the utility surface. The utility surface shown is for the function

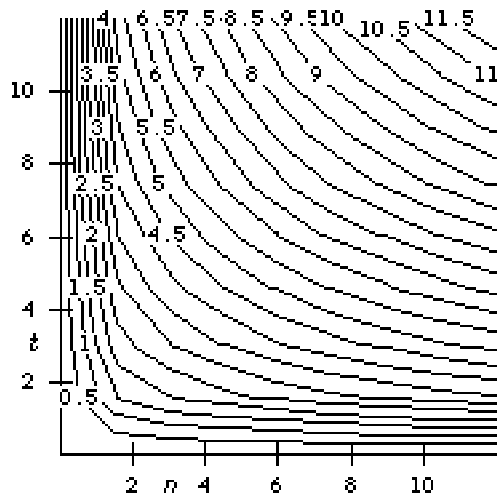
$$u(c_0, c_1) = c_0^{0.4} c_1^{0.6}.$$

figure 3: Contour Lines for the utility function



points, i.e., consumption bundles, which have the same elevation, i.e., utility; these lines are known as indifference curves. These intertemporal indifference curves represent the individual's preferences for consumption now versus then. The absolute value of the slope of these curves at any consumption pair yields the individual's **intertemporal marginal rate of substitution**, subsequently denoted as **mrs**, **measures the value of consumption now in terms of consumption then**. A steeper indifference curve corresponds to greater "impatience" for consumption now. The indifference curves in the next figure also illustrate the notion of a decreasing marginal rate of substitution, i.e., as the individual increases consumption now the value of consumption now in terms of consumption then decreases.

figure 4: Indifference Curves



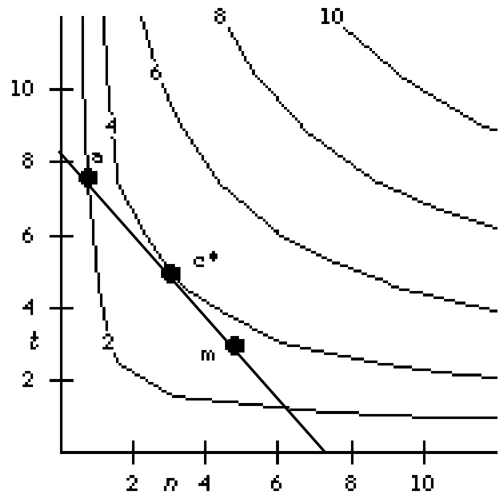
The individual's choice problem may be characterized as a constrained maximization problem, i.e., select the consumption bundle to

$$\text{maximize } u(c_0, c_1)$$

$$\text{subject to } c_0 + \frac{c_1}{1+r} = m_0 + \frac{m_1}{1+r}$$

The solution to this problem is demonstrated in the next figure. Note that at the consumption bundle c*, the individual's marginal rate of substitution equals one plus the rate of interest, i.e., **mrs* = 1 + r** .

figure 5: the optimal consumption bundle



Equivalently, at c*, an indifference curve is tangent to the budget line. This consumption bundle is clearly optimal and so the optimality condition is mrs = 1 + r. The optimality condition has a simple interpretation; it says that at the margin c* the individual values consumption now in terms of consumption then at its opportunity cost. As further evidence that mrs = 1 + r is the optimal condition, consider the consumption pair denoted by a = (a_1, a_2) in the figure. Letting mrs^a denote the marginal rate of substitution there, notice that $mrs^a > 1 + r$. This inequality says that the individual values consumption now in terms of consumption then at more than its opportunity cost. Hence, the individual can increase utility by increasing consumption now.

Exercises:

[1] Provide a sketch to show how the budget line changes given: (i) an increase in income now; (ii) an increase in income then; (iii) an increase in the interest rate r.

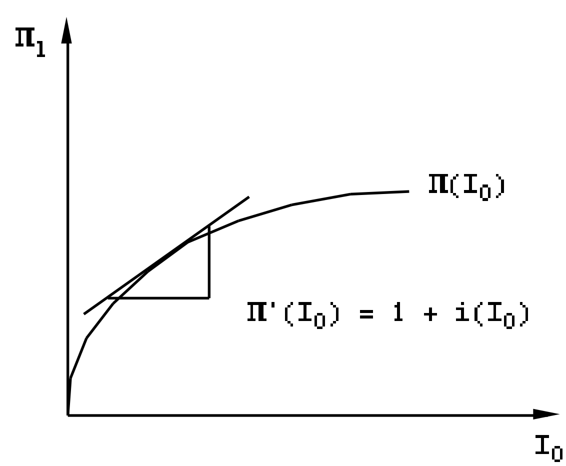
[2] Suppose the individual has intertemporal preferences specified by $u(c_0, c_1) = \min\{c_0, c_1\}$. Sketch the indifferences curves and show the optimal consumption bundle.

[3] Suppose the individual has the intertemporal preferences specified in the last problem and an income pair such that $m_0 > m_1$. Does the individual lend or borrow in the financial market? How does the lending or borrowing decision change given an increase in the interest rate?

Case II: The Investment Decision

Next suppose the individual may restructure the time pattern of consumption by investing in capital goods market but not by saving in the financial market. Let I_0 denote the dollar investment in capital goods and Π_1 denote the total dollar return on the investment; let $\Pi_1 = \Pi(I_0)$ and suppose $\Pi(I_0)$, i.e., the investment frontier, is a function which increases at a decreasing rate in the dollar investment. The function Π is shown in the following figure. The slope of the function

figure 6: the investment frontier



at a point is $\Pi'(I_0) \equiv 1 + i(I_0)$, where i is an interest rate called the marginal efficiency of investment. Since the payoff Π_1 increases at a decreasing rate, the marginal efficiency of investment also decreases as I_0 increases.

The Net Present Value and Internal Rate of Return [5]

Although the firm proprietor is not concern with the net present value, or equivalently, the net future value, of the investment project, it is appropriate at this point to identify the investment level which maximizes the net present value. Let npv and nfv denote net present and future value, respectively. Then

$$npv(I_0) = - I_0 + \frac{\Pi_0(I_0)}{1 + r}$$

and

$$nfv(I_0) = - (1 + r)\, I_0 + \Pi(I_0).$$

Maximizing npv and nfv, of course, yields the same investment level. The derivative of net future value with respect to the investment level is

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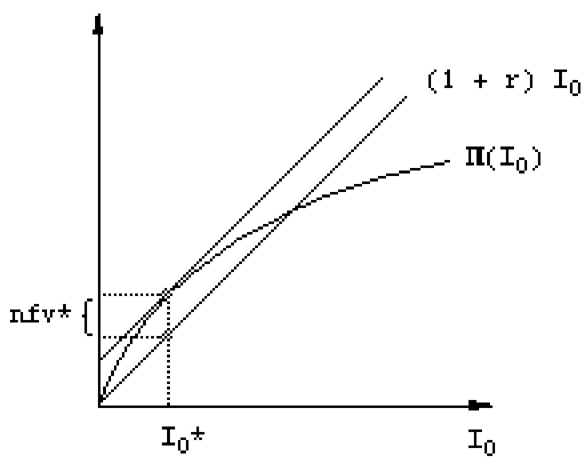
[5] Having introduced the investment frontier, it is appropriate to consider the npv and IRR concepts at this point. This discussion is separated from the analysis of case two to emphasize that the proprietor does not even consider the net present value concept and cannot because the firm has no access to a financial market in this case.

$$\frac{dnfv}{dI_0} = -(1 + r) + \Pi'(I_0) = -(1 + r) + (1 + i(I_0)).$$

At the investment level which maximizes nfv, this derivative is zero and so $(1 + i(I_0^*)) = (1 + r)$, or equivalently, $\mathbf{i(I_0^*) = r}$. This condition simply says that the last dollar invested must yield the same rate of return as is available in the financial market. The investment I_0^* is shown in the following figure. Note that the vertical distance between $\Pi(I_0)$ and $(1 + r)\, I_0$ is the nfv and I_0^* maximizes this distance.

figure 7: the investment level which maximizes npv and nfv



An alternative method of finding the optimal investment utilizes iso-net future value lines. To construct an iso-nfv line hold the net future value fixed at nfv^0. Then all of the (I_0, Π_1) pairs which yield this net future value are implicitly defined by

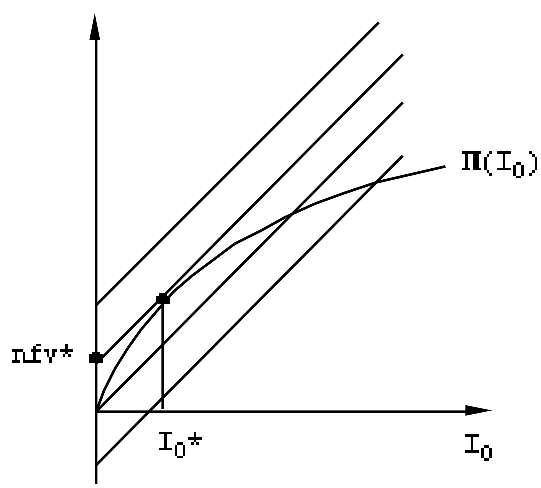
$$nfv^0 = -(1 + r)\, I_0 + \Pi_1,$$

or, explicitly, by

$$\Pi_1 = nfv^0 + (1 + r)\ I_0.$$

Note that nfv^0 is the intercept and (1 + r) is the slope of the iso-nfv line. The following figure shows a whole family of iso-nfv lines. It also shows that the investment pair which maximizes nfv is on the highest iso-nfv line. Then the intercept of this highest iso-nfv line is the maximum net future value.

figure 8: iso-nfv lines and the optimal investment



It is also possible to provide a graphical interpretation of the internal rate of return, i.e., IRR, on the project. Recall that the internal rate of return is implicitly defined as that rate of return which yields a zero net present value, or equivalently, a zero net future value. Hence, the $IRR(I_0)$ is implicitly defined by the condition

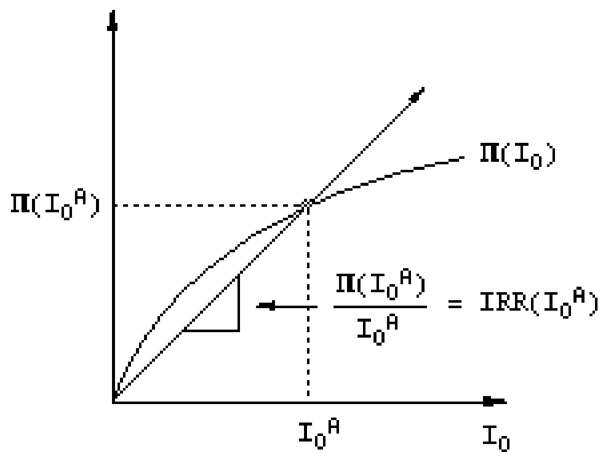
$$nfv(I_0) = 0 = -\ (1 + IRR(I_0))\ I_0 + \Pi(I_0),$$

or equivalently, by the condition

$$1 + IRR(I_0) = \frac{\Pi(I_0)}{I_0} \quad .$$

This shows that one plus the internal rate of return can be interpreted graphically as the slope of a cord from the origin to a point on the investment frontier, as shown in the following figure.

figure 9: the internal rate of return



Note that if, as assumed, $\Pi(I_0)$ increases at a decreasing rate then the internal rate of return decreases in I_0.

Now, consider the proprietor's decision problem. As before let (c_0, c_1) and (m_0, m_1) denote the consumption and income pairs. Of course, this income pair excludes the investment in capital goods. Once an investment decision is made a new income pair $(m_0$

$- I_0$, $m_1 + \Pi(I_0)$) is generated. Since the individual cannot participate in the financial market, it follows that consumption now and then are

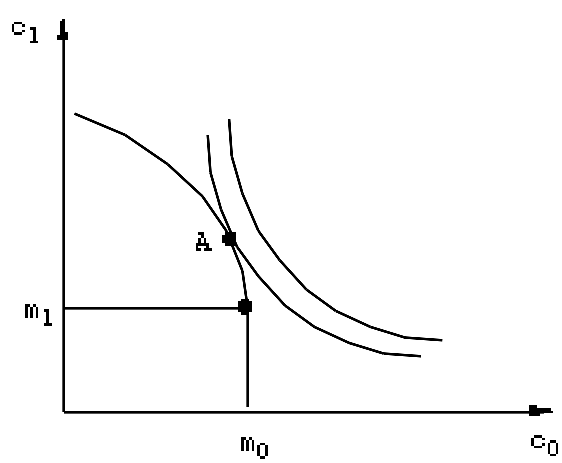
$$c_0 = m_0 - I_0,$$

and,

$$c_1 = m_1 + \Pi(I_0),$$

respectively. Note that the investment decision determines consumption now and then. The proprietor's optimal decision may be characterized by sketching the investment frontier in consumption space, as shown in the following figure.

figure 10: the proprietor's choice given no access to the financial market



The constraint on the proprietor's choice is the investment frontier represented in the consumption space. The proprietor selects an income pair on this frontier to which none is preferred. Point A in the figure represents the optimal choice for the manager, i.e., given the proprietor's preferences which are contained in the indifference curves. Note, at A the indifference curve is tangent to the frontier. Hence, the proprietor's condition for an optimal choice is **$mrs^A = 1 + i(I_0^A)$**.[6] **This optimality condition says that the value of more consumption now in terms of consumption then equals the opportunity cost of more consumption now** .

Exercises:

[4] Let the investment frontier be specified as $\Pi_1 = \min\{\kappa I_0, M\}$, where κ and M are positive constants. Sketch this frontier. Also provide a sketch of the marginal efficiency of investment and the internal rate of return.

[5] Show that $nfv(I_0) > 0$ implies $IRR(I_0) > r$ and that $nfv(I_0) < 0$ implies $IRR(I_0) < r$.

[6] Suppose $\Pi' > 0$ and $\Pi'' < 0$. Show that $i < IRR$ for all positive investment levels.[7]

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[6] To generate this optimal condition, maximize $u(m_0 - I_0, m_1 + \Pi(I_0))$ with respect to I_0. The first order condition for this maximization problem is

$$\frac{\partial u}{\partial c_0}\frac{\partial c_0}{\partial I_0} + \frac{\partial u}{\partial c_1}\frac{\partial c_1}{\partial I_0} = \frac{\partial u}{\partial c_0}(-1) + \frac{\partial u}{\partial c_1}\Pi'(I_0^A) = 0,$$

or equivalently,

$$mrs^A \equiv \frac{\frac{\partial u}{\partial c_0}}{\frac{\partial u}{\partial c_1}} = \Pi'(I_0^A) = 1 + i(I_0^A).$$

[7] Exercises denoted by a require some use of the calculus.

[7] Provide a sketch of an investment frontier which would yield a negative net present value for any positive level of investment.

[8] In case II, show that investing in a negative net present value project can be optimal.

Case III: The Optimal Investment and Savings Decisions

Now let the individual not only invest in capital goods market but also save in the financial market. If I_0 dollars are invested now and generate $\Pi(I_0)$ dollars then, it follows that the present value of the income stream is

$$m_0 - I_0 + \frac{m_1 + \Pi(I_0)}{(1 + r)},$$

or equivalently,

$$m_0 + \frac{m_1}{(1 + r)} - I_0 + \frac{\Pi(I_0)}{(1 + r)}.$$

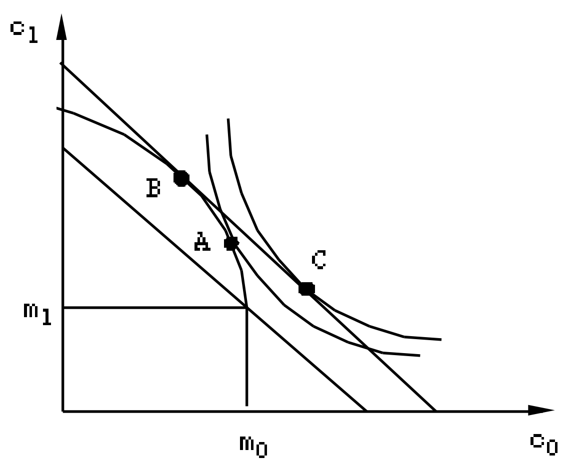
Now, the budget line in the financial market must be rewritten so that the present value of the income stream is represented. Hence, given the investment choice (I_0, $\Pi(I_0)$), the budget line is

$$c_0 + \frac{c_1}{(1 + r)} = m_0 + \frac{m_1}{(1 + r)} - I_0 + \frac{\Pi(I_0)}{(1 + r)}.$$

Again, this is simply the condition that the present value of consumption equals the present value of income. Now, however, the income pair is $(m_0 - I_0, m_1 + \Pi(I_0))$. The position of the individual's budget line depends the investment decision because that decision alters the income pair.

The individual has two roles. One of the roles is as the proprietor of the firm. In that capacity the individual selects the investment level. The other role is as a consumer. In this capacity, the individual selects the pair (c_0, c_1), or equivalently, a savings level. The feasible investment and consumption decisions are represented by the investment frontier and the associated budget line, respectively, in the following figure.

figure 11: the optimal investment and consumption decisions



The individual, in the role of proprietor, selects the investment level indicated implicitly by the point B. Then, the individual, in the role of consumer, selects the consumption bundle indicated by the point C. Note that, **at B the condition $1 + i(I_0^B) = 1 + r$ holds**, while **at C the condition $mrs^C = 1 + r$ holds**. Also, observe that the condition

$\mathbf{i(I_0^B) = r}$ does not depend on the individual's preferences and that it is the condition for a maximum net present value.

An alternative intuitive explanation is as follows: The individual has preferences consistent with the observation "more is preferred to less." The individual selects the investment plan (I_0, $\Pi(I_0)$) which maximizes the present value of total income because by doing so the individual obtains the highest possible budget line in the financial market. To see this, note that the budget line for any investment decision intersects the horizontal axis at

$$m_0 + \frac{m_1}{(1 + r)} - I_0 + \frac{\Pi(I_0)}{(1 + r)} =$$

$$m_0 + \frac{m_1}{(1 + r)} + npv(I_0)$$

and the highest trading line maximizes this value and so it yields the greatest capability for consumption now and then. This is the Fisher Separation Result, i.e., all individuals, irrespective of their preference for consumption now versus then, select the same investment plan. Maximizing the present value of income is equivalent to maximizing the net present value of the investment.

Recall that the analysis was not begun with the objective of maximizing the net present value. The objective was to maximize the consumer\proprietor's utility and this yielded the result that any consumer\proprietor makes the investment decision to maximize net present value. Hence, the roles of proprietor and saver can be separated.

Exercises:

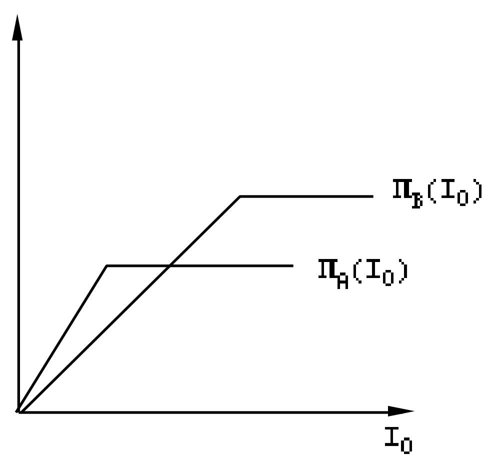
[9] Why is it reasonable to say that the financial market rate of return r is the cost of capital?

[10] Sketch the case in which any positive investment in a project yields a negative npv and show that the consumer\proprietor chooses not to invest.

[11] Show how the proprietor's investment choice is affected by a reduction in the interest rate r.

[12] Suppose that the proprietor selects an investment level either greater than or less than the level I_0^B shown in figure 11. Show that, in either case, the individual is worse off and that this result does not depend on whether the individual is a borrower or lender.

[13] Suppose the proprietor can invest in one of two mutually exclusive projects. Let Π_A and Π_B denote the investment frontiers for the two projects so that $\Pi_A(I_0)$ is the revenue generated in period one if project A is selected and $\Pi_B(I_0)$ if projected B is selected. The investment frontiers are shown in the following figure.



Show and explain the following:

(i) The Fisher Separation result holds in terms of which project is selected as well as the scale at which the project is operated.

(ii) Specify the conditions under which project B will be selected over project A.

(iii) Which project has the larger internal rate of return (IRR)? If you select project A or B on the basis of which has the larger IRR, is your choice consistent with your analysis in (ii)?

[14] Find an investment level in figure eight which yields a negative net future value. Show how the IRR for that investment compares with the financial rate of return.

The Multiperiod Fisher Model

The time horizon of the model may be extended by letting c_t and m_t denote consumption and income, respectively, at dates t = 0, 1, 2, ..., T. The individual still selects a consumption plan to which none is preferred in the budget constraint. Now, however, there is more than one interest rate. Let $r_{\tau t}$ denote the interest rate for borrowing or lending between the dates τ and t. If the time horizon is T = 2, then there are three rates of interest, i.e., r_{01}, r_{02}, and r_{12} where, for example, r_{01} is the rate between zero and one. In the absence of arbitrage opportunities, the budget line is

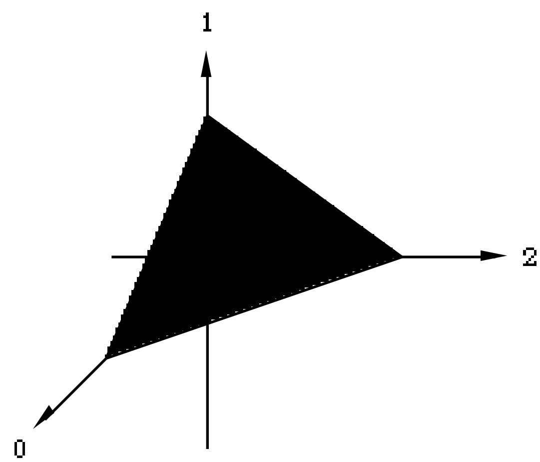
$$\sum_{t=0}^{t=2} \frac{c_t}{(1 + r_{0t})^t} = \sum_{t=0}^{t=2} \frac{m_t}{(1 + r_{0t})^t}$$

and the interpretation is the same, i.e., the present value of consumption equals the present value of income. Notice that only the rates r_{0t} are used here in calculating the present values. This is valid because of the following **no arbitrage condition** :

$$(1 + r_{02})^2 = (1 + r_{01})\,(1 + r_{12}).$$

It may be observed that the no arbitrage condition holds if all the rates are the same. If this no arbitrage condition holds then the budget constraint is as shown in the following figure. If it does not hold then, quite naturally, there is no budget constraint because all individuals can become indefinitely wealthy through arbitrage.

figure 12: the multiperiod budget constraint



To see this arbitrage opportunity, suppose

$$(1 + r_{02})^2 < (1 + r_{01})\,(1 + r_{12}).$$

To arbitrage the financial markets suppose the individual borrows long-run and lends short-run. In particular, suppose one dollar is borrowed for two periods; this commits the individual to pay $(1 + r_{02})^2$ at $t = 2$. Next, lend the dollar at $t = 0$ to obtain $(1 + r_{01})$ dollars at $t = 1$. Similarly, lend $(1 + r_{01})$ dollars at $t = 1$ to obtain $(1 + r_{01})(1 + r_{12})$ dollars at $t = 2$. The net future value of these transactions is

$$nfv = (1 + r_{01})(1 + r_{12}) - (1 + r_{02})^2 > 0.$$

This is the arbitrage opportunity. Note that the individual used borrowed money and made money on it. Also, note that there is no limit to this process. This implies that the inequality cannot survive in the financial markets and, in this case, the long-run rate must rise relative to the two short-run rates.